Aligned electric and magnetic Weyl fields

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Abstract. We analyze the spacetimes admitting a direction for which the relative electric and magnetic Weyl fields are aligned. We give an invariant characterization of these metrics and study the properties of its Debever null vectors. The directions 'observing' aligned electric and magnetic Weyl fields are obtained for every Petrov type. The results on the no existence of purely magnetic solutions are extended to the wider class having homothetic electric and magnetic Weyl fields.

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1. Introduction

A notable number of known solutions of Einstein equations have been obtained by imposing restrictions on the algebraic structure of the Weyl tensor. Thus, wide families of algebraically special solutions have been found by considering coordinates or frames adapted to the multiple Debever direction that these spacetimes admit. Nevertheless, there is a lack of knowledge about algebraically general solutions, and they have usually been obtained by considering spacetime symmetries. One way to correct this situation is to consider subclassifications of algebraically general spacetimes and look for solutions in every defined class.

Some classifications of type I metrics involving first order differential Weyl concomitants have been proposed (see [1] and references therein). Nevertheless, the more basic restrictions that can be imposed on the Weyl tensor are the algebraic ones. Debever [2] suggested a classification of type I spacetimes based on the nullity of a Weyl invariant scalar. A similar kind of conditions are satisfied in some classes of 'degenerate' type I spacetimes defined by McIntosh and Arianhood [3]. They used the adimensional complex scalar $M = \frac{a^3}{b^2} - 6$, $a = \text{Tr}\,\mathcal{W}^2$ and $b = \text{Tr}\,\mathcal{W}^3$ being, respectively, the quadratic and the cubic Weyl symmetric scalar invariants. The scalar M is related to the Penrose cross-ratio invariant [4] and it governs the geometry defined by the Debever null directions: $M = 0$ in Petrov type D and, in type I, M is real positive or infinite when the four Debever directions span a 3–plane [3]; the case M real negative occurs

when the Penrose-Rindler [5] disphenoid associated with the Debever directions has two equal edges [6]. Elsewhere [7] we have presented an alternative approach to analyzing this Debever geometry using the complex angle between the principal bivectors and the unitary Debever bivectors. Here we show that the case M negative can be reinterpreted in terms of permutability properties with respect to the metric tensor of a frame [8] built with the Debever null vectors. This result was presented without proof at the Spanish Relativity Meeting 1998 [9].

The electric and magnetic Weyl tensors are gravitational quantities E and B attached to any observer and playing an analogous role to the electric and magnetic fields [10] [11]. Some classes of spacetimes can be defined by imposing that an observer exists for which the electric and magnetic parts of the Weyl tensor satisfy some restriction. Thus, we have the purely electric ($B = 0$) and the purely magnetic ($E = 0$) spacetimes. These conditions depend on the observer and, consequently, they are no invariant a priory. Nevertheless, McIntosh *et al.* [12] showed that the Weyl-electric and Weyl-magnetic spacetimes admit an intrinsic characterization in terms of some scalar invariants: M must be real positive or infinite, and a must be real, positive in the electric case and negative in the magnetic case. Consequently, the Debever directions of a type I purely electric and purely magnetic Weyl tensor span a 3–plane. Elsewhere [13] we have generalized the purely electric and purely magnetic concepts by considering electric and magnetic Weyl parts with respect to an arbitrary direction. These generalized Weyl-electric or Weyl-magnetic spacetimes also permit the scalar M to take real negative values, the cubic scalar b being real or purely imaginary, respectively. Thus, the new classes of gravitational fields that we have considered in [13] admit a partially symmetric frame built with Debever vectors.

All the results quoted above show that the spacetimes with the invariant M being a real function have Debever directions with special properties, and the subfamily of them with b^2 a real function can be identified in terms of the electric and magnetic Weyl fields. Is it possible to give a characterization of the other metrics for which M is a real function by using the relative electric and magnetic fields? In this paper we give un affirmative answer to this question by showing that the necessary and sufficient conditions for M to be real is that the spacetime admits a (non necessarily time-like) direction for which the electric and magnetic parts are aligned (they are proportional tensors). This kind of condition has been already considered for the time-like case [14] [3] and we analyze here a generic causal character. Moreover we determine for every Petrov type all the directions (without restriction on its causal character) for which the relative electric and magnetic fields are aligned.

The properties imposed on the electric and magnetic Weyl fields imply integrability conditions which some times are very restrictive. Thus, whereas a lot of physically interesting purely electric solutions are known, severe restrictions appear in dealing with purely magnetic ones (see references in [13] [15]). We want to remark here that there are no vacuum solutions with a purely magnetic type D Weyl tensor [16], and McIntosh *et al.* [12] have conjectured that a similar restriction could take place for a

wide class of type I spacetimes.

We can find in the literature significant steps in supporting the McIntosh *et al.* conjecture. It was shown for a shear-free observer [14] [17] and, recently, Van der Bergh has showed the conjecture provided that the observer defines a normal congruence [18] or for a freely falling observer [19]. Moreover, the conjecture is also true under weaker conditions on the shear and vorticity tensors that trivially hold when the shear or the vorticity vanish [15]. These results are also valid for non vacuum solutions with vanishing Cotton tensor [15].

It is worth pointing out that an extension of the conjecture is known for type D metrics. Indeed, elsewhere [20] we have shown that not only the purely magnetic solutions are forbidden, but also a wider class of type D solutions. More precisely, we have shown [20]: *if a spacetime with vanishing Cotton tensor has a type D Weyl tensor with (complex) eigenvalues of constant argument, then it is a purely electric solution.* This means that the constant argument takes, necessarily, the values 0 or π.

A question arises in a natural way: is there a similar generalization for algebraically general spacetimes?, that is, is it possible to extend to a wider class of type I solutions the restrictions that one has obtained for the purely magnetic ones? This is, precisely, the extension that we show in the present paper: when a type I metric with vanishing Cotton tensor has homothetic electric and magnetic Weyl fields with respect to an observer satisfying the kinematic restrictions quoted above (those obtained in [15]), then the spacetime is purely electric.

The article is organized as follows. In section 2 we present the basic formalism and we define the concepts of Weyl-aligned spacetime and Weyl-aligned direction. In section 3 we determine, for every Petrov type, the Weyl aligned directions. In section 4 we characterize the full class of Weyl-aligned metrics intrinsically, as well as, some specific subclasses, and we analyze in detail the type I Weyl-aligned spacetimes by studying the properties of the Debever null vectors. Finally, section 5 is devoted to extend the results on the no existence of purely magnetic solutions to the spacetimes with homothetic electric and magnetic fields.

2. Weyl-aligned spacetimes

Let (V_4, g) be an oriented and time-oriented spacetime of signature $\{-, +, +, +\}$, and let W be its Weyl tensor. We can associate to any unitary vector field v ($v^2 = \epsilon$, $\epsilon = \pm 1$) the electric and magnetic Weyl fields:

$$E = E[v] \equiv W(v; v), \qquad B = B[v] \equiv *W(v; v) \qquad (1)$$

where $*$ is the Hodge dual operator and we denote $W(v; v)_{\alpha\gamma} = W_{\alpha\beta\gamma\delta}v^\beta v^\delta$. The electric and magnetic fields (1) with respect to a spacelike or timelike congruence determine the Weyl tensor fully. This fact was pointed out years ago for the timelike case [10, 11], and also holds for a spacelike congruence [13]. When v is a null vector we can also define the electric and magnetic fields (1) but, in this case, they do not determine the Weyl tensor

[13]. Nevertheless, here we also consider the electric and magnetic parts with respect to a null direction. In this work we will use the following definitions.

Definition 1 *A metric is Weyl-aligned at a point of spacetime when there is a vector v for which the relative electric and magnetic Weyl fields are aligned at this point. Then, the angle $\sigma \in [0, \pi[$ such that $\cos \sigma B[v] + \sin \sigma E[v] = 0$ is named rotation index associated with v.*

Definition 2 *We say that v is a Weyl-aligned vector if the relative electric and magnetic Weyl fields are aligned for some rotation index σ.*

These definitions extend the concepts of generalized Weyl-electric and Weyl-magnetic spacetimes and Weyl-electric and Weyl-magnetic directions given in [13]. The purely electric (resp. magnetic) case corresponds to the rotation index to take the value 0 (resp. $\pi/2$). On the other hand, the rotation index has the following interpretation: the Weyl tensor W writes

$$W = \cos \sigma W_0 + \sin \sigma * W_0 \tag{2}$$

W_0 being a Weyl-like tensor which is purely electric for the vector v. That is, σ plays the role of a duality rotation.

From here we work in the self-dual complex formalism. A self–dual 2–form is a complex 2–form $\mathcal{F}$ such that $*\mathcal{F} = \mathrm{i}\mathcal{F}$. We can associate biunivocally to every real 2–form F the self-dual 2–form $\mathcal{F} = \frac{1}{\sqrt{2}}(F - \mathrm{i} * F)$. We here refer to a self–dual 2–form as a *bivector*. The endowed metric on the 3-dimensional complex space of the bivectors is $\mathcal{G} = \frac{1}{2}(G - \mathrm{i}\,\eta)$, G being the usual metric on the 2–form space, $G = \frac{1}{2}g \wedge g$, $(g \wedge g)_{\alpha\beta\mu\nu} = 2(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})$, and η being the metric volume element. A $\mathcal{G}$-unitary bivector $\mathcal{U} = \frac{1}{\sqrt{2}}(U - \mathrm{i} * U)$ corresponds to every timelike unitary simple 2–form U $((U, *U) = 0, (U, U) = -1)$, and $\mathcal{H} = \frac{1}{\sqrt{2}}(H - \mathrm{i} * H)$ is a null bivector for $\mathcal{G}$ when H is singular $((H, H) = (H, *H) = 0)$.

A unitary bivector $\mathcal{U}$ defines a timelike 2–plane with volume element U and its orthogonal spacelike 2–plane with volume element $*U$. We denote these *principal 2–planes* as their volume element. The null directions $l_\pm$ in the 2–plane U are the (real) eigendirections of $\mathcal{U}$ and they are called *principal directions*. These principal directions may be parameterized in such a way that $U = l_- \wedge l_+$. On the other hand a null bivector $\mathcal{H}$ define two null *fundamental 2–planes*, with volume elements H and $*H$, which cut in the unique (real) eigendirection l that $\mathcal{H}$ admits. Just a parametrization of the null vector l exists such that it is future-pointing and $H = l \wedge e_2$, where e_2 is a spacelike unitary vector orthogonal to l, and fixed up to change $e_2 \hookrightarrow e_2 + \mu l$. With this parametrization we name l *fundamental vector* of $\mathcal{H}$.

The algebraic classification of the Weyl tensor W can be obtained by studying the traceless linear map defined by the self–dual Weyl tensor $\mathcal{W} = \frac{1}{2}(W - \mathrm{i} * W)$ on the bivectors space. We can associate to the Weyl tensor the complex scalar invariants

$$a \equiv \operatorname{Tr} \mathcal{W}^2 = \rho_1^2 + \rho_2^2 + \rho_3^2, \qquad b \equiv \operatorname{Tr} \mathcal{W}^3 = \rho_1^3 + \rho_2^3 + \rho_3^3 = 3\rho_1\rho_2\rho_3 \tag{3}$$

where ρ_i are the eigenvalues. It will be also useful to consider the adimensional scalar invariant [12, 3]:

$$M \equiv \frac{a^3}{b^2} - 6 = \frac{2(\rho_1 - \rho_2)^2(\rho_2 - \rho_3)^2(\rho_3 - \rho_1)^2}{9\rho_1^2\rho_2^2\rho_3^2} \tag{4}$$

The invariant M is well defined for the Petrov types I, D or II if we permit it to be infinite in the case of a type I metric with $b = 0$. In types D and II, M is identically zero, and we extend its validity by considering that it also takes the null value for type N and type III metrics.

In terms of the invariants a and b the characteristic equation reads $x^3 - \frac{1}{2}ax - \frac{1}{3}b = 0$. Then, Petrov-Bel classification follows taking into account both the eigenvalue multiplicity and the degree of the minimal polynomial. The algebraically regular case (type I) occurs when $6b^2 \neq a^3$ and so the characteristic equation admits three different roots. If $6b^2 = a^3 \neq 0$, there are a double root and a simple one and the minimal polynomial distinguishes between types D and II. Finally, if $a = b = 0$ if all the roots are equal and so zero, and the Weyl tensor is of type O, N or III, depending on the degree of the minimal polynomial.

The electric and magnetic Weyl fields (1) associated to a unitary vector field v give, respectively, the real and imaginary parts of the Petrov matrix $\mathcal{W}(v; v)$:

$$2\mathcal{W}(v; v) = W(v; v) - \mathrm{i} *W(v; v) \equiv E[v] - \mathrm{i}\, B[v] \tag{5}$$

On the other hand, there are four scalars built with the electric and magnetic Weyl fields which are independent, up to sign, of the unitary vector v ($v^2 = \epsilon$) [10, 11]. In fact they are the real and imaginary parts of the complex scalar invariants a and b:

$$a = (\mathrm{Tr}\, E^2 - \mathrm{Tr}\, B^2) - 2\,\mathrm{i}\,\mathrm{Tr}(E \cdot B), \tag{6}$$

$$b = -\epsilon[(\mathrm{Tr}\, E^3 - 3\,\mathrm{Tr}(E \cdot B^2)) + \mathrm{i}(\mathrm{Tr}\, B^3 - 3\,\mathrm{Tr}(E^2 \cdot B))] \tag{7}$$

3. Weyl-aligned directions

Here we determine for every Petrov type: (i) the conditions for the spacetime to be Weyl-aligned, (ii) the rotation index σ for which this condition holds, and (iii) the Weyl-aligned vectors corresponding to every rotation index σ. We will express these vectors in terms of σ and the canonical frames or other geometric elements associated with the Weyl tensor. As the richness of these frames depends on the Petrov type [21], we will consider every algebraic class separately.

In every case, we summarize the spacetime geometry that the Weyl *canonical bivectors* [21] determine, and we remark the relationship with the canonical frames that they define. In order to determine the Weyl-aligned directions we do not need to solve any equation because we can use the results obtained in [13] on the Weyl electric directions. Indeed, taking into account that σ gives the duality rotation (2), W_0 being purely electric, we have the following:

Lemma 1 *The necessary and sufficient condition for v to be a Weyl-aligned vector for W with associated rotation index σ is v to be a Weyl-electric vector for $W_0 = \cos\sigma W - \sin\sigma * W$, that is, $e^{-\mathrm{i}\sigma}\mathcal{W}(v;v)$ to be real.*

Thus, we can use the results of [13] by changing $\mathcal{W}$ by $e^{-\mathrm{i}\sigma}\mathcal{W}$ in every Petrov type.

3.1. Type N

In Petrov type N, a unique null bivector $\mathcal{H}$ exists such that the self-dual Weyl tensor may be written [21]

$$\mathcal{W} = \mathcal{H} \otimes \mathcal{H} \tag{8}$$

The *canonical bivector* $\mathcal{H}$ determines the *fundamental 2–planes* H and $*H$ and the *fundamental vector* l (which determines the quadruple Debever direction) of a type N Weyl tensor.

From (8), we have that $e^{-\mathrm{i}\sigma}\mathcal{W} = \left(e^{-\mathrm{i}\frac{\sigma}{2}}\mathcal{H}\right) \otimes \left(e^{-\mathrm{i}\frac{\sigma}{2}}\mathcal{H}\right)$. Then, from lemma 1, the condition that a vector v must satisfy to be Weyl-aligned with associated rotation index σ follows from the Weyl-electric solutions in [13] replacing H by $\cos\frac{\sigma}{2}H - \sin\frac{\sigma}{2}*H$. Thus we have:

Proposition 1 *Every type N spacetime is Weyl-aligned, the rotation index σ being arbitrary. The Weyl-aligned vectors with associated rotation index σ are those on the planes $\cos\frac{\sigma}{2}H \pm \sin\frac{\sigma}{2}*H$. These Weyl-aligned vectors are the fundamental vector l (which satisfies $E[l] = B[l] = 0$) and the other (spacelike) vectors lying on the null 3–plane orthogonal to l.*

3.2. Type III

In type III, a unitary bivector $\mathcal{U}$ and a null bivector $\mathcal{H}$ exist such that the self dual Weyl tensor may be written [21]

$$\mathcal{W} = \mathcal{U} \,\widetilde{\otimes}\, \mathcal{H} \tag{9}$$

The *canonical null bivector* $\mathcal{H}$ determines the *fundamental 2–planes* H and $*H$ and the *fundamental vector* l (which determines the triple Debever direction). The canonical bivectors $\mathcal{H}$ and $\mathcal{U}$ define an oriented and ortochronous null real frame $\{l, l', e_2, e_3\}$ such that $U = \pm\, l \wedge l'$, $H = l \wedge e_2$. This frame is characterized by l to be the triple Debever direction, l' the simple one, and e_2 (resp. e_3) to be the intersection of the planes $*U$ and H (resp. $*H$) [21].

Now, $e^{-\mathrm{i}\sigma}\mathcal{W} = \mathcal{U} \,\widetilde{\otimes}\, \left(e^{-\mathrm{i}\sigma}\mathcal{H}\right)$, and from lemma 1, the condition that a vector v must satisfy to be Weyl-aligned with associated rotation index σ follows from the Weyl-electric solutions in [13] replacing H by $\cos\sigma H - \sin\sigma * H$. Thus we have:

Proposition 2 *Every type III spacetime is Weyl-aligned, the rotation index σ being arbitrary. The Weyl-aligned vectors with associated rotation index σ are the triple Debever direction l (which satisfies $E[l] = B[l] = 0$) and the spacelike direction $\cos\sigma e_3 - \sin\sigma e_2$.*

3.3. Type D

The self-dual Weyl tensor of a Petrov type D spacetime takes the canonical form

$$\mathcal{W} = 3\rho\,\mathcal{U} \otimes \mathcal{U} + \rho\mathcal{G} \tag{10}$$

where $\mathcal{U}$ is the *canonical bivector* and $\rho = -\frac{b}{a}$ is the double eigenvalue [21]. Thus, in this case, the *principal 2–planes* U and $*U$ are outlined. The principal directions $l_\pm$ of U are the double Debever directions that the type D admits.

We have $e^{-i\sigma}\,\mathcal{W} = 3e^{-i\sigma}\rho\,\mathcal{U} \otimes \mathcal{U} + e^{-i\sigma}\rho\mathcal{G}$ and, taking into account lemma 1, the conditions for $\mathcal{W}$ to be Weyl-aligned follows from the results in [13] on the purely electric type D metrics, just replacing ρ by $e^{-i\sigma}\rho$ and imposing this last expression to be real. But it means that σ must be either the argument θ of the Weyl eigenvalue or $\theta - \pi$ (when θ is bigger than π). This way, we get

Proposition 3 *Every type D spacetime is Weyl-aligned and the rotation index is $\sigma = \theta \pmod{\pi}$, θ being the argument of a Weyl eigenvalue. The Weyl-aligned vectors are the principal ones, that is, $v \in U$ or $v \in *U$. The only null Weyl-aligned directions are the double Debever directions $l_\pm$. In the 2–plane U there timelike and spacelike Weyl-aligned directions. Every $v \in *U$ is a spacelike Weyl-aligned direction.*

3.4. Type II

The self-dual Weyl tensor of a Petrov type II spacetime takes the canonical form

$$\mathcal{W} = 3\rho\,\mathcal{U} \otimes \mathcal{U} + \rho\mathcal{G} + \mathcal{H} \otimes \mathcal{H} \tag{11}$$

$\mathcal{U}$ being the unitary eigenbivector associated to the simple eigenvalue $\rho = -\frac{b}{a}$, and $\mathcal{H}$ being the only eigendirection associated to the double eigenvalue. These geometric elements define an oriented and orthochronous null real frame $\{l, l', e_2, e_3\}$ such that $U = \pm\, l \wedge l'$ and $H = l \wedge e_2$. This frame is defined by l to be the double Debever direction, and e_2 (resp. e_3) to be the intersection of the 2–planes $*U$ and H (resp. $*H$) [21].

From (11), we have that $e^{-i\sigma}\,\mathcal{W} = 3e^{-i\sigma}\rho\,\mathcal{U} \otimes \mathcal{U} + e^{-i\sigma}\rho\mathcal{G} + e^{-i\frac{\sigma}{2}}\mathcal{H} \otimes e^{-i\frac{\sigma}{2}}\mathcal{H}$ and, taking into account lemma 1, the results in [13] can be applied just changing ρ by $e^{-i\sigma}\rho$ and $\mathcal{H}$ by $e^{-i\frac{\sigma}{2}}\mathcal{H}$. Thus, we have:

Proposition 4 *Every type II spacetime is Weyl-aligned and the rotation index is $\sigma = \theta \pmod{\pi}$, θ being the argument of a Weyl eigenvalue. The Weyl-aligned vectors are the double Debever direction l and the spacelike directions $\cos\frac{\theta}{2}e_2 + \sin\frac{\theta}{2}e_3$ and $-\sin\frac{\theta}{2}e_2 + \cos\frac{\theta}{2}e_3$.*

3.5. Type I

The self–dual Weyl tensor of a type I spacetime takes the canonical form:

$$\mathcal{W} = -\sum_{j=1}^{3} \rho_j\,\mathcal{U}_j \otimes \mathcal{U}_j \tag{12}$$

where $\{\mathcal{U}_j\}$ are the unitary eigenbivectors associated with the simple eigenvalues ρ_j [21]. The *canonical bivectors* $\mathcal{U}_i$ define six *principal 2–planes* U_i and $*U_i$ which cut in the four orthogonal *principal directions* that a type I metric admits. The unitary principal vectors define the Weyl canonical frame frame $\{e_\alpha\}$ that satisfies $U_i = e_0 \wedge e_i$ [7].

Thus, $e^{-\mathrm{i}\sigma}\mathcal{W} = -\sum e^{-\mathrm{i}\sigma}\rho_j\, \mathcal{U}_j \otimes \mathcal{U}_j$ and, taking into account lemma 1, the results of [13] can be applied changing ρ_j by $e^{-\mathrm{i}\sigma}\rho_j$. These three complex numbers are real if, and only if, the ratio between two eigenvalues is real or infinity, the argument of every one being either σ or $\pi + \sigma$. On the other hand, $e^{-\mathrm{i}\sigma}\rho_j$ are complex conjugated for two values of j if, and only if, the two eigenvalues ρ_j have the same modulus, the argument θ of the third eigenvalue being either σ or $\pi + \sigma$. So, we have

Proposition 5 *A type I spacetime is Weyl-aligned if, and only if, one of the two following conditions hold:*

(1) The ratio between every two eigenvalues is real (or infinity).

(2) Two of the eigenvalues have the same modulus.

If condition (1) holds, then the rotation index is $\sigma = \theta$ (mod π), θ being the argument of a Weyl eigenvalue. Moreover, the Weyl-aligned directions are the timelike Weyl principal direction e_0 and the spacelike Weyl principal directions e_i.

If condition (2) holds and the third eigenvalue has different modulus, say $|\rho_1| = |\rho_2| \neq |\rho_3|$, then the rotation index is $\sigma = \theta_3$ (mod π), θ_3 being the argument of the eigenvalue ρ_3. Moreover, the Weyl-aligned directions are the spacelike directions $e_1 \pm e_2$.

If condition (2) holds and we have equimodular eigenvalues, then there are three rotation index given by $\sigma_i = \theta_i$ (mod π), θ_i being the argument of every Weyl eigenvalue. Moreover, the Weyl-aligned directions with associated rotation index σ_i are the spacelike directions $e_j \pm e_k$, i, j, k taking different values.

4. Some classes of Weyl-aligned spacetimes

Once the rotation index and the Weyl-aligned directions have been found for an arbitrary Weyl tensor by considering the different Petrov-Bel types, in this section we study and characterize same classes of Weyl-aligned spacetimes. We begin by considering some direct consequences of the results in previous section.

Corollary 1 *If a metric is Weyl-aligned for a timelike or a null direction with associated rotation index σ, then it is Weyl-aligned for a spacelike direction with the same rotation index.*

A null direction is Weyl-aligned if, and only if, it is a multiple Debever direction. Consequently, a spacetime is Weyl-aligned for a null direction if, and only if, it is algebraically special.

Every timelike Weyl-aligned direction is a Weyl principal direction.

On the other hand, we also recover the following result suggested by Barnes [14]:

Proposition 6 *If a spacetime is Weyl-aligned for a timelike direction, then the Weyl tensor is Petrov-Bel type I, D or O, and this direction is a Weyl principal one.*

We have also shown that an algebraically special spacetime is always Weyl-aligned, but the more degenerate the Petrov type is, the more richness in the number of rotation index exists. More precisely, we have:

Proposition 7 *Every algebraically special spacetime is Weyl-aligned.*

For Petrov types N and III ($a = b = 0$) the rotation index σ is arbitrary. For Petrov types D and II ($a^3 = 6b^2 \neq 0$) the rotation index is $\sigma = \theta$ (mod π), θ being a Weyl eigenvalue.

From here we will analyze in detail the Weyl-aligned type I spacetimes and we will show that they can be characterized in terms of the properties of their Debever directions. From this study we will get the following invariant characterization of the Weyl-aligned spacetimes:

Theorem 1 *A spacetime is Weyl-aligned if, and only if, the Weyl invariant scalar M defined in (4) is real. Moreover:*

(i) $M = 0$ if, and only if, the spacetime is algebraically special.

(ii) $M > 0$ if, and only if, the spacetime is Petrov type I and it is Weyl-aligned for a timelike direction; this direction is the principal one and the metric is also Weyl-aligned for the three spacelike principal directions.

(iii) $M < 0$ if, and only if, the spacetime is Petrov type I and it is Weyl-aligned for the bisectors $e_i \pm e_j$ of a spacelike principal 2-plane.

Let us now consider a Type I Weyl tensor. We have already shown [7] that, for every Weyl eigenvalue, say ρ_3, we can consider the unitary bivectors $\mathcal{V}_\epsilon$, $\epsilon = \pm 1$:

$$\mathcal{V}_\epsilon = \cos \Omega \, \mathcal{U}_1 + \epsilon \, \sin \Omega \, \mathcal{U}_2 \tag{13}$$

where the complex Weyl invariant Ω is given by

$$\cos 2\Omega = \frac{3\rho_3}{\rho_2 - \rho_1} \tag{14}$$

The bivectors $\mathcal{V}_\epsilon$ are *unitary Debever bivectors* [21] , that is, their principal directions are the four simple Debever directions that a type I Weyl tensor admits.

These expressions have been obtained privileging ρ_3. A similar argument with the other two eigenvalues lead to other pairs of Debever bivectors and gives us other angles Ω_1 and Ω_2. These angles are not independent, and from (14) it is easy to show that

$$\cos^2 \Omega_1 = \frac{1}{\sin^2 \Omega}, \qquad \cos^2 \Omega_2 = -\tan^2 \Omega \tag{15}$$

Witting $\Omega = \phi - i\psi$, we can calculate the principal directions of the bivectors (13), and we obtain the following expression for the Debever directions [7]:

$$l_{\epsilon\pm} = \cosh \psi e_0 \pm \cos \phi e_1 \pm \epsilon \sin \phi e_2 + \epsilon \sinh \psi e_3, \qquad (\epsilon = \pm 1) \tag{16}$$

On the other hand, taking into account (14), the invariant M given in (4) can be expressed in terms of Ω. So, for every M, this expression poses a cubic equation for $\cos 2\Omega$, being every solution associated with one of the angles Ω_i quoted in (15). More precisely, we have for $k = 0, 1, 2$:

$$\cos 2\Omega = \sqrt{3}\left\{N + \frac{\beta}{2}(N - \mathrm{i})\left[\beta\, e^{\frac{2\pi k}{3}\mathrm{i}} + e^{\frac{-2\pi k}{3}\mathrm{i}}\right]\right\}, \quad \beta \equiv \sqrt[3]{\frac{N + \mathrm{i}}{N - \mathrm{i}}}, \quad N \equiv \sqrt{\frac{6}{M}} \quad (17)$$

Let go now on the Weyl-aligned type I metrics. We start with the first subclass pointed out in Proposition 5: the ratio between every two eigenvalues is real (or infinity). This case implies that the M given in (4) is real positive (or infinity), and this condition leads to $\cos 2\Omega$ be a real function if we take into account (17). But if $\cos 2\Omega$ is real, the ratio between two eigenvalues is real (or infinity) as a consequence of (14). Thus, we have three equivalent conditions.

On the other hand, if $\Omega = \phi - \mathrm{i}\psi$, $\cos 2\Omega$ is real when $\mathrm{senh}\psi\, \cos\phi\, \mathrm{sen}\phi = 0$. But if we take into account the expression (16), this condition states that the four Debever directions are linearly dependent and they span the 3–plane orthogonal to e_j, ρ_j being the shortest eigenvalue accordingly with (14). We can summarize these results that complete those of McIntosh et al. [3, 12] (see also [7]) as:

Theorem 2 *In a type I spacetime the following statements are equivalent:*

(i) *The metric is Weyl-aligned for a principal direction (and then for every principal direction).*

(ii) M *is real positive or infinite.*

(iii) $\cos 2\Omega$ *is real.*

(iv) *The ratio between every two eigenvalues is real (or infinite).*

(v) *The Debever directions span a 3–plane.*

Moreover, if one of the above conditions hold, the 3-plane that Debever directions span is orthogonal to e_j, ρ_j being the shortest eigenvalue. The case $M = \infty$ corresponds to $b \equiv tr\mathcal{W}^3 = 0$.

Let us consider the second subclass in Proposition 5: two of the eigenvalues have the same modulus. This means that the ratio between these two eigenvalues lies on the unit circle, $\frac{\rho_2}{\rho_1} = e^{i\theta}$, $\theta \in (0, 2\pi)$, and then the invariant M given in (4) is real negative or infinity. This condition implies that one of the solutions in (17) is a purely imaginary function or zero. But if $\cos 2\Omega_3$ is purely imaginary or zero, the eigenvalues ρ_2 and ρ_1 have the same modulus as a consequence of (14). Thus, we have three equivalent conditions and, taking into account proposition 5, we have established a similar result to the four first statements of the previous theorem.

Now we look for a description of this case in terms of the Debever directions. Elsewhere [6] the Penrose-Rindler [4] disphenoid has been used for this purpose. Nevertheless we interpret here this case in terms of permutability properties of a frame built with the Debever null vectors.

When the Debever directions $\{l_a\}_{a=1}^4$ are independent, they become a null frame. It is told that two vectors $\{l_1, l_2\}$ of a null frame are permutable (or that the frame is P_2) if $(l_1, l_b) = (l_2, l_b)$ $(b = 3, 4)$, that is, if we can not distinguish between l_1 and l_2 making the product with the other two vectors [8]. A remarkable property is that if a null frame is P_2, we can reparameterize the vectors of the frame to make permutable the other two directions too, that is, we can get a $P_2 \times P_2$ frame. In the same way, it is told that all the vectors are permutable (or that the frame is P_4) if all the products (l_a, l_b) $(a \neq b)$ are equal [8].

From (16) it is easy to show that if the Debever directions are independent, they admit a reparametrization to a $P_2 \times P_2$ frame if, and only if, $\cos 2\Omega$ is purely imaginary. As we have seen, this means that there exists two of the eigenvalues, say ρ_1 and ρ_2 $(\rho_1 \neq \pm\rho_2)$ having the same modulus. Then, the pair of permutable Debever directions are those which are the principal directions of V_ϵ constructed privileging ρ_3, that is the principal directions of the Debever bivectors such that their bisectors are $\mathcal{U}_1$ and $\mathcal{U}_2$.

The particular case of the three eigenvalues having the same modulus leads to $M = -6$ $(a = 0)$ and the three solutions of (17) are $\cos \Omega_k = \sqrt{3}$i. Moreover then, and only then, a reparametrization of the Debever vectors exists such that we can built a P_4-frame. All these results can be summarized as:

Theorem 3 *In a type I spacetime the following statements are equivalent:*

(i) *The metric is Weyl-aligned for a direction which is not a principal one.*

(ii) *M is real negative or infinity.*

(iii) *$\cos 2\Omega$ is purely imaginary or zero.*

(iv) *There exist two eigenvalues such that its ratio lies in the unit circle (have the same modulus).*

(v) *The frame of Debever vectors can be reparameterized to be $P_2 \times P_2$.*

Moreover, if ρ_1 and ρ_2 have the same modulus, the pairs of permutable vectors are the principal directions of the Debever bivectors given in (13). In this case V_ϵ and $*V_{-\epsilon}$ cut each other in the bisectors $e_1 \pm e_2$ which are the Weyl-aligned directions of point (i).

The frame of Debever vectors can be reparameterized to be P_4 if, and only if, all the eigenvalues have the same modulus, that is when $M = -6$ $(a = 0)$. In this case the metric is Weyl-aligned for the bisectors $e_i \pm e_j$ of every spacelike principal plane.

5. Homothetic electric and magnetic Weyl fields in vacuum: kinematic restrictions

Several results are known that restrict the existence of purely magnetic spacetimes. From the initial one by Hall [16] which showed that there no purely magnetic type D vacuum solutions, some works are known that extend this result in different ways. In one hand, the extension for Type I metrics conjectured by McIntosh *et al.* [12] has been shown when the observer is: (i) shear-free [14] [17], (ii) vorticity-free [18], (iii) geodesic

[19]. The vorticity-free and shear-free conditions have been weakened recently [15] by means of first-order differential conditions which hold trivially when $\sigma = 0$ or $\omega = 0$. In this last work another kind of progression is acquired: the restriction is also valid for non vacuum solutions with vanishing Cotton tensor. This extension has been also shown for type D spacetimes in a paper [20] where a third kind of generalization is obtained: not only the purely magnetic solutions are forbidden, but also those whose Weyl eigenvalue has a constant argument different of 0 or π.

In this section we will give a similar extension for type I spacetimes. Indeed, as we have shown in section 3, every type D metric is Weyl-aligned and the rotation index is given by the argument of the Weyl eigenvalue. Thus, the extension for type D spacetimes quoted above applies when, for an observer, the electric and magnetic Weyl fields satisfy $E = kB$, k being a constant factor. Now we generalize the kinematic restrictions obtained in [15] to the type I spacetimes with this *homothetic* property. We start by giving the following

Definition 3 *We will say that the electric and magnetic Weyl fields E and B with respect to an observer u are homothetic if they are aligned with a constant rotation index σ, that is, $\cos \sigma B + \sin \sigma E = 0$, $\mathrm{d}\sigma = 0$.*

The case $\sigma = \pi/2$ corresponds to the purely magnetic case which has been analyzed in [15]. We will show now that homothetic spacetimes are subjected to similar restrictions on the kinematic coefficients of the observer that the purely magnetic ones.

Under the hypothesi of a vanishing Cotton tensor, the Bianchi identities take the same expression than in the vacuum case [15]. Thus they may be written in the 1+3 formalism [22]:

(i) $\operatorname{div} E = -3\ B(\omega) + [\sigma, B]$

(ii) $\operatorname{div} B = -\ [\sigma, E] + 3\ E(\omega)$

(iii) $\hat{\dot{E}} - \operatorname{curl} B = -\theta\ E + 3\ E \hat{\times} \sigma -\ \omega \wedge E + 2\ \ a \wedge B$

(iv) $\hat{\dot{B}} + \operatorname{curl} E = -\theta B + 3\ \sigma \hat{\times} B - \omega \wedge B - 2\ a \wedge E$

where D is the covariant spatial derivative, div and curl are, respectively, the covariant spatial divergence and curl operators, $\wedge$ and $[\ ,\ ]$ are the generalized covariant vector products and $\hat{\ }$ means the projected trace-free symmetric part (see for example [22] for more details). Now, if σ is constant and $\sigma \neq 0$, then $E = -\cot \sigma B$, and removing E from the equations above, a straightforward calculation leads to:

$$[\sigma, B] = 3B(\omega) \tag{18}$$

$$\operatorname{div} B = 0 \tag{19}$$

$$\operatorname{curl} B = -2a \wedge B \tag{20}$$

But these are the same restrictions that we have used in [15] for the $E = 0$ case. Then, taking into account the results in [15], we can state:

Theorem 4 *In a spacetime with vanishing Cotton tensor if the electric and magnetic fields are homothetic with respect to an observer u satisfying one of the following conditions:*

(i) $\mathrm{Tr}(\mathrm{curl}\,\sigma)^2 - 3\,\mathrm{Tr}(\hat{D}\omega + 2a\widehat{\otimes}\omega)^2 \neq 2(\mathrm{curl}\,\sigma, \hat{D}\omega + 2a\widehat{\otimes}\omega)$

(ii) $\mathrm{Tr}(\hat{D}\omega + 2a\widehat{\otimes}\omega)^2 = 0$

(iii) $\mathrm{Tr}(\hat{D}\omega + 2a\widehat{\otimes}\omega)^2 \geq \mathrm{Tr}(\mathrm{curl}\,\sigma)^2$

Then, the spacetime is purely electric and u is a Weyl principal direction.

From here, a corollary follows.

Corollary 2 *In a spacetime with vanishing Cotton tensor if the electric and magnetic fields are homothetic with respect to a shear-free or a vorticity-free observer u, then the spacetime is purely electric and u is a Weyl principal direction.*

The result which states that the vacuum solutions with electric and magnetic Weyl fields proportional for a shear-free observer are, necessarily, purely electric has been also presented recently by Barnes [23].

Acknowledgments

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